# 6.2 Linear and Almost Linear Systems

revisit a system from last time:

$$x' = -2x - y \qquad \text{cp}: (0,0)$$
$$y' = -x - 2y \qquad \text{nodal sink}$$

a related system:

$$x' = -2x - y + 5 \qquad \text{cp}: (2,1)$$
$$y' = -x - 2y + 4 \qquad \text{nodal sink}$$

the phase diagrams are identical but centered at different cp's.

why? $x' = -2x - y$
$y' = -x - 2y$ $\rightarrow$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$x' = -2x - y + 5$
$y' = -x - 2y + 4$ cp: $(2, 1)$

define $u = x - 2$ $v = y - 1$

then $u' = x'$ $v' = y'$

and $x = u + 2$, $y = v + 1$

$u' = -2(u+2) - (v+1) + 5$

$v' = -(u+2) - 2(v+1) + 4$

simplify

$$u' = -2u - v \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$
$$v' = -u - 2v \rightarrow$$

the same system
as the homogeneous case
each centered the their
respective origin.

$\vec{x}' = A\vec{x}$ A has eigenvalues $r_1, r_2$

critical pt is asymptotically stable

if $r_1 < 0$, $r_2 < 0$ or $r_1, r_2$ have negative real part stable if real part of $r_1, r_2$ is zero

unstable if one of $r_1$ or $r_2$ (or both) is positive or if $r_1, r_2$ have positive real part

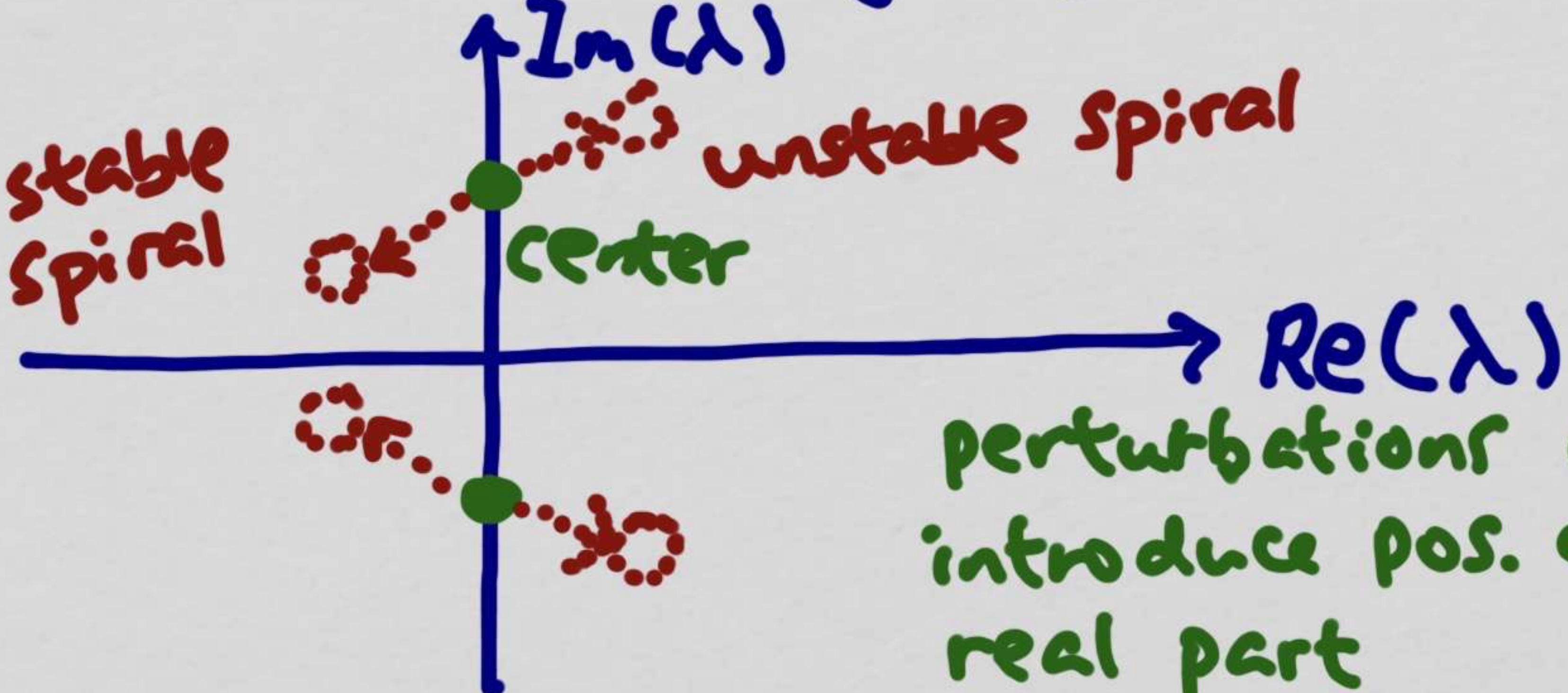
How sensitive are these to small perturbations to the elements of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

It turns out that if the cp is asympt. stable or unstable, then

Small perturbations do NOT affect stabity and the trajectories (node/spiral/etc) do not change.

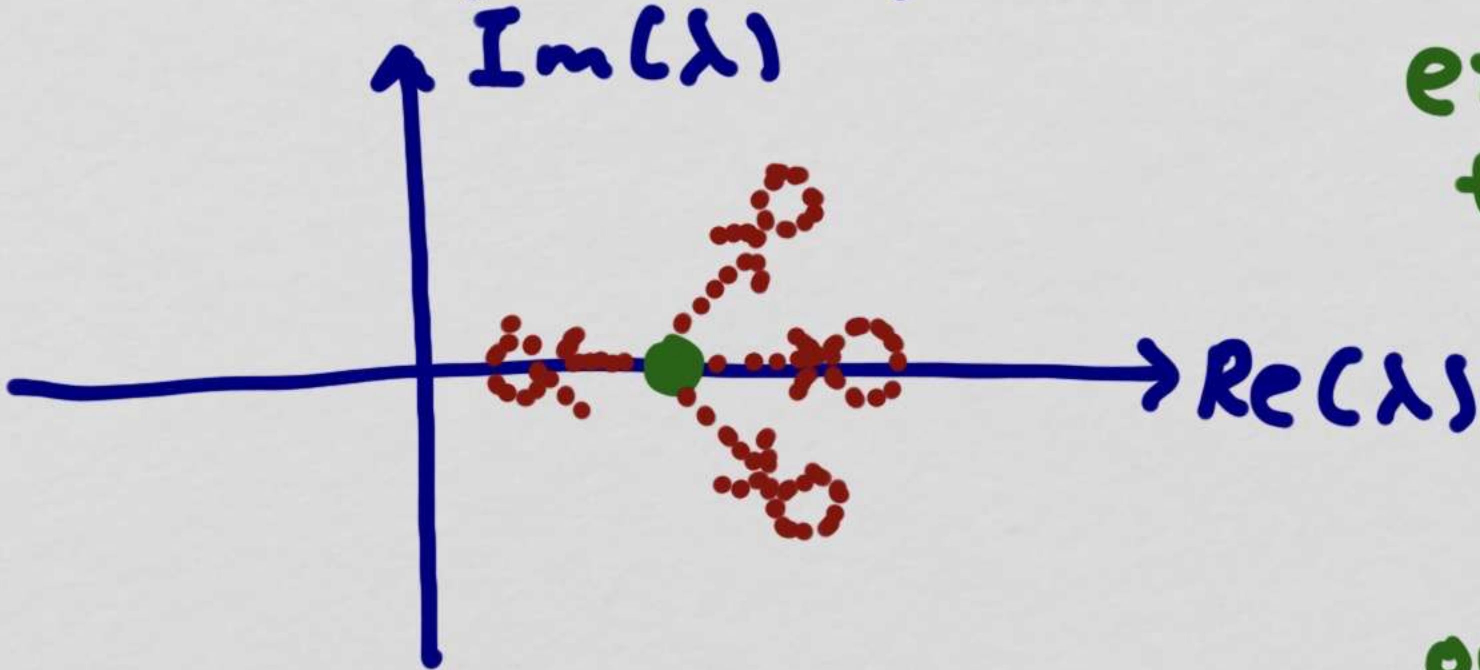
There are two cases where the stability and trajectories can change.

Case 1: purely imaginary eigenvalues




perturbations can introduce pos. or neg. real part

# Case 2 : repeated eigenvalues




either split into two distinct but of the same sign as original or introduce imaginary part while keeping the real part with the same sign

Stability does <u>NOT</u> change but the trajectories might

Nonlinear systems have phase diagrams that look like those of linear systems near each critical pt.

this means $x' = F(x,y)$

$$y' = G(x,y)$$

$$\Big\downarrow \text{ near each cp}$$

$$\vec{x}\,' = A\vec{x} + \vec{g}(\vec{x})$$

if $(x_0, y_0)$ is a cp

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|\vec{g}(\vec{x})|}{|\vec{x}|} = 0$$

then the system is almost linear or locally linear for example, $x' = -x + xy$

$y' = -2y + 8xy$

cp: $(0,0)$, $(\frac{1}{4}, 1)$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} xy \\ 8xy \end{bmatrix}}_{\vec{g}(\vec{x})}$$

notice $(x,y) \to (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{|\vec{g}|}{|\vec{x}|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{65}\, xy}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{65}\,xy}{\sqrt{x^2+y^2}}$$

let $r=\sqrt{x^2+y^2}$

then $(x,y)\to(0,0)$ is the same as $r\to 0$

$x = r\cos\theta,\ y = r\sin\theta$

$$\lim_{r\to 0} \frac{\sqrt{65}\, r^{\cancel{2}} \cos\theta\sin\theta}{\cancel{r}} = 0$$

so, this is an almost linear system near $(0,0)$

it will behave like $\vec{x}\,' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \vec{x}$ near $(0,0)$

$\lambda = -1, -2$

asymp. stable node

now look at near $(\frac{1}{4}, 1)$ what is $A$?

$$x' = -x + xy \qquad \text{cp}: (\tfrac{1}{4}, 1)$$

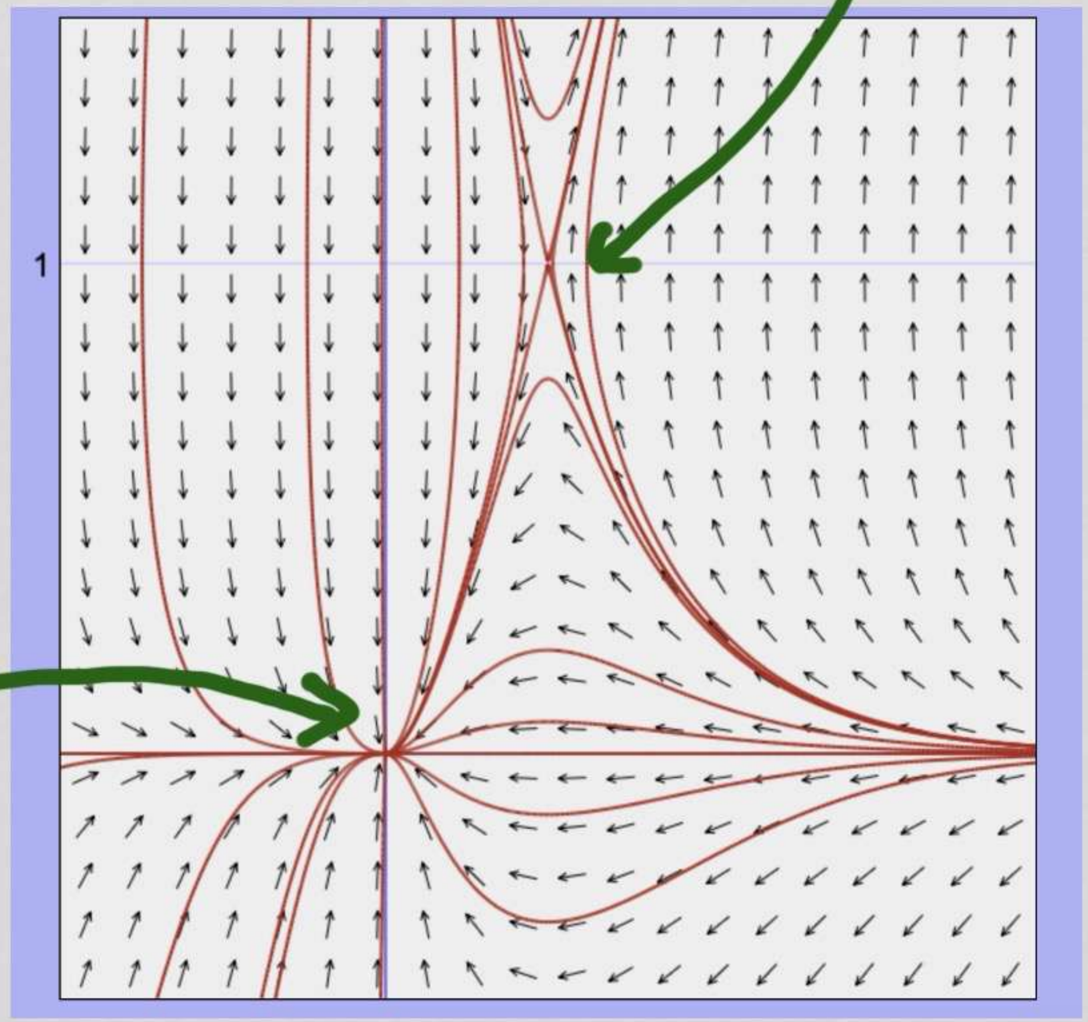
$$y' = -2y + 8xy \qquad u = x - \tfrac{1}{4}, \; v = y - 1$$

$$u' = \tfrac{1}{4} v + uv$$
$$v' = 8u + 8uv \rightarrow \begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{4} \\ 8 & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} uv \\ 8uv \end{bmatrix}$$

$\lambda = \pm\sqrt{2}$ saddle pt

neither saddle pt or asympt. stable node is sensitive to perturbation

Saddle pt




asympt. stable node